Solution of two mode bosonic Hamiltonians and related physical systems

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We have constructed the quasi-exactly-solvable two-mode bosonic realizations of $su(2)$ and $su(1,1)$ algebra. We derive the relations leading to the conditions for quasi-exact-solvabilty of two-boson Hamiltonians by determining a general procedure which maps the Schwinger representations of the $su(2)$ and $su(1,1)$ algebras to the Gelfand-Dyson representations respectively. This mapping allows us to study nonlinear quantum optical systems in the framework of quasi-exact-solvabilty. Our approach also leads to a simple construction of special functions of two variables which are the most appropriate functions to study quasi-probabilities in quantum optics.

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INTRODUCTION

The exact solutions of the multi-bosons systems play an important role in quantum mechanics. The Hamiltonians are employed in quantum optics most often involving multi-bosons. During last decade a great attention has been paid to examine different quantum optical models with Hamiltonians given by multi-bosons[1–5]. The quantum optical systems whose Hamiltonians expressed by Casimir operator of the symmetry group can exactly be solved[6]. The quantum optical systems with Hamiltonians including nonlinear functions in Lie algebra generators can be analyzed by using mainly numerical methods, because Lie algebraic techniques are non-efficient for such systems and most of the other analytical techniques require in general tedious calculations.

In quantum mechanics there exist physical systems whose finite part of spectrum can be exactly obtained in closed forms. They are known as quasi-exactly-solvable(QES) systems[7–11, 14]. Dating back over fifteen years there has been a great deal of interest in QES systems. The one dimensional QES systems based on sl_2-algebra have been classified by Turbiner[7]. The usual approach to the analysis of the one dimensional QES systems is to express the Hamiltonian of the corresponding physical system as linear and bilinear combination of the generators of the sl_2-algebra. The necessary conditions for the normalizabilty of the wavefunction of the QES systems based on sl_2- algebra were completely determined in[8]. The QES models, either in the forms of differential equations or of the single boson Hamiltonian, have been extended by Dolya and Zaslavskii[12, 13].

The aim of this paper is to extend the QES systems to the multi-boson systems and to obtain the solution of the multi-boson Hamiltonians, in particular, Hamiltonians of the nonlinear quantum optical systems. The formalism presented here also leads to the construction of special functions with two variables which are appropriate to study many quantum optical problems and quasi-probabilty[15, 16]. We think that the QES bosonic systems deserve special treatments for they have many applications in physics. The suggested approach can be extended to the matrix multi-bosonic systems.

In the Bosonic Hamiltonian, it is more convenient to use the bosonic representations of the $su(2)$ and $su(1,1)$ generators and determine the conditions of quasi-exact solvability. The single boson realization of the $su(1,1)$ algebra has been studied in[12, 13]. In this work we follow a different strategy to obtain the conditions for QES of the two-bosonic systems.

The algebras of the groups $SU(2)$ and $SU(1,1)$ have useful applications in quantum physics[17–19]. They are used to generate the energy spectra while the representation matrices of the group could be used to calculate time dependent excitations of the bound states and the scattering states respectively[6]. As usual the group elements and their associated algebras can be expressed in terms of bosonic operators.

This paper is organized as follows. The constructions of the two-mode bosonic realization of the $su(2)$ and $su(1,1)$ algebra are briefly reviewed in section 2. In section 3, we propose the transformation of the boson creation and annihilation operators leads to the two class $su(2)$ and two class $su(1,1)$ one dimensional realizations that is useful to study QES systems. The one dimensional realizations of the $su(2)$ and $su(1,1)$ algebras are presented by similarity transformation section 4. We also discuss the some applications of our approach. We demonstrate the procedure

presented here can be used to solve a rich family of physical systems. In particular, the Karassiov-Klimov Hamiltonian, Hamiltonians of the second and third generation harmonic oscillator systems, Hamiltonian of the quantal systems under thermal effect have been solved in section 5. We also demonstrate the relations between QES one-dimensional Schrödinger equation and two-mode bosonic Hamiltonians leads to the construction of the two dimensional differential equations of the special functions, in this section. The paper ends with a brief discussion and conclusion.

CONSTRUCTION OF TWO-BOSON HAMILTONIANS: $su(2)$ AND $su(1,1)$ ALGEBRAS

A convenient way[6] to construct a spectrum generating algebra for systems with a finite number of bound states is by introducing a set of boson creation and annihilation operators. We introduce two boson operator, a_1 and a_2, obey the usual commutation relations

$$[a_1, a_1] = \left[a_1, a_2^+\right] = \left[a_2, a_2^+\right] = 0, \quad \left[a_1, a_1^+\right] = \left[a_2, a_2^+\right] = 1. \tag{1}$$

The bilinear combinations $a_1^+a_1, a_1^+a_2, a_2^+a_1$ and $a_2^+a_2$ generates the group $U(2)$ and $a_1^+a_1$, a_1a_2, $a_1^+a_2^+$ and $a_2^+a_2$ generates the group $SU(1,1)$. Let us start by introducing three generators of $SU(2)$,

$$J_+ = a_1^+a_2, \quad J_- = a_2^+a_1, \quad J_0 = \frac{1}{2}\left(a_1^+a_1 - a_2^+a_2\right). \tag{2}$$

These are the Schwinger representation of $SU(2)$ algebra and they satisfy the commutation relations

$$[J_+, J_-] = 2J_0 \quad [J_0, J_\pm] = \pm J_\pm \tag{3}$$

The fourth generator is the total boson number operator

$$N = a_1^+a_1 + a_2^+a_2 \tag{4}$$

which commutes with the $SU(2)$ generators. The Casimir operator of this structure is given by

$$J = J_-J_+ + J_0(J_0+1) = \frac{1}{4}N(N+2). \tag{5}$$

If we denote the eigenvalues of the operator J by

$$J = j(j+1) \tag{6}$$

the irreducible representations of $SU(2)$ are characterized by the total boson number

$$j = N/2 \tag{7}$$

where $N = 0, 1, 2, \ldots$.

The Schwinger representation of $su(1,1)$ algebra can be constructed by considering bosonic realizations of the generators:

$$K_+ = a_1^+a_2^+, \quad K_- = a_2a_1, \quad K_0 = \frac{1}{2}\left(a_1^+a_1 + a_2^+a_2 + 1\right). \tag{8}$$

They satisfy the commutation relations

$$[K_+, K_-] = -2K_0 \quad [K_0, K_\pm] = \pm K_\pm. \tag{9}$$

The total boson number operator of this algebra is given by

$$L = a_1^+ a_1 - a_2^+ a_2, \tag{10}$$

and it commutes with the generators of the $su(1,1)$ algebra. The Casimir operator of this structure can be expressed in terms of number operator, such that

$$C = -J_- J_+ + J_0(J_0+1) = \frac{1}{4}(L+1)(L-1). \tag{11}$$

It is obvious that number operator and Casimir operator commute. If the eigenvalues of the operator C is denoted by

$$C = \ell(\ell+1) \tag{12}$$

the irreducible representations of $su(1,1)$ algebra are characterized by a total boson number

$$L = -(2\ell+1). \tag{13}$$

For the exactly solvable case the Hamiltonian have been written in terms of Casimir invariants of the algebra. In this case the eigenvalues and eigenfunction of the Hamiltonian can be obtained in the closed form. Our task is now to obtain the conditions for QES of the $su(2)$ and $su(1,1)$ algebra given in (2) and (8). The conditions can be obtained by connecting two-mode bosonic realizations of the $su(2)$ and $su(1,1)$ algebra to the $sl(2,R)$ and $sl_2(R)$ algebra, respectively. Since Casimir operator J commutes with N, and C commute with L the linear combinations of the generators of each algebra can be diagonalized within the representation $[N]$ and $[L]$, respectively. The abstract boson algebra can be associated with the exactly soluble Schrödinger equations by using the differential operator realizations of boson operators. This connection opens the way to an algebraic treatment of a large class of potentials of practical interest.

TRANSFORMATION OF THE BOSONIC OPERATORS

In the previous section we have summarized the construction of the two-mode bosonic realizations of the $su(2)$ and $su(1,1)$ algebras. In this section we develop a procedure to transform the generators in the Schwinger representation to a more suitable representation to determine its relation with the QES systems. This can be done by various methods. Here we follow a different strategy to obtain the connection between the two-mode bosonic systems and QES systems. Let us introduce the following similarity transformation induced by the operator

$$S = (a_2^+)^{\alpha a_1^+ a_1} \tag{14}$$

where α is a constant and in order to obtain Gelfand-Dyson representations of the $su(2)$ and $su(1,1)$ algebra it will be constrained to ± 1. In general, it will be shown later, it is not necessary to set $\alpha = \pm 1$, to construct QES systems. This property allow us to study a wide range of physical systems. The operator S acts on the state $|n_1, n_2\rangle$ as follows,

$$S\,|n_1, n_2\rangle = (a_2^+)^{\alpha n_1}\,|n_1, n_2\rangle = \sqrt{\frac{n_2!}{(n_2+\alpha n_1)!}}\,|n_1, n_2+\alpha n_1\rangle\,. \tag{15}$$

Since a_1 and a_2 commute, the transformation of a_1 and a_1^+ under S can be obtained by writing $a_2^+ = e^b$, with $[a_1, b] = [a_1^+, b] = 0$,

$$\begin{aligned} S a_1^+ S^{-1} &= e^{\alpha b a_1^+ a_1} a_1^+ e^{-\alpha b a_1^+ a_1} = a_1^+ (a_2^+)^{\alpha} \\ S a_1 S^{-1} &= e^{\alpha b a_1^+ a_1} a_1 e^{-\alpha b a_1^+ a_1} = a_1 (a_2^+)^{-\alpha} \end{aligned} \tag{16}$$

and transformation of a_2 and a_2^+ as follows

$$\begin{aligned} Sa_2^+S^{-1} &= (a_2^+)^{\alpha a_1^+ a_1} a_2^+ (a_2^+)^{-\alpha a_1^+ a_1} = a_2^+ \\ Sa_2 S^{-1} &= (a_2^+)^{\alpha a_1^+ a_1} a_2 (a_2^+)^{-\alpha a_1^+ a_1} = a_2 - \alpha a_1^+ a_1 (a_2^+)^{-1}. \end{aligned} \tag{17}$$

In a similar manner, from Schwinger representation to the Gelfand-Dyson representation which is suitable to study QES systems, the generators of $su(2)$ and $su(1,1)$ algebra in the Schwinger representation can be transformed by introducing the following operator:

$$T = a_2^{\alpha a_1^+ a_1} \tag{18}$$

The operator T acts on the two-boson state as

$$T\left|n_1, n_2\right\rangle = a_2^{\alpha n_1}\left|n_1, n_2\right\rangle = \sqrt{\frac{n_2!}{(n_2-\alpha n_1)!}}\left|n_1, n_2-\alpha n_1\right\rangle. \tag{19}$$

Since a_1 and a_2 commute, the transformation of a_1 and a_1^+ under S can be obtained by letting $a_2 = e^c$ with $[a_1, c] = [a_1^+, c] = 0$,

$$\begin{aligned} Ta_1^+T^{-1} &= e^{\alpha c a_1^+ a_1} a_1^+ e^{-\alpha c a_1^+ a_1} = a_1^+ (a_2)^{\alpha} \\ Ta_1 T^{-1} &= e^{\alpha c a_1^+ a_1} a_1 e^{-\alpha c a_1^+ a_1} = a_1 (a_2)^{-\alpha} \end{aligned} \tag{20}$$

The transformation of a_2 and a_2^+ are as follows:

$$\begin{aligned} Ta_2^+T^{-1} &= a_2^{\alpha a_1^+ a_1} a_2^+ a_2^{-\alpha a_1^+ a_1} = a_2^+ + \alpha a_1^+ a_1 a_2^{-1}. \\ Ta_2 T^{-1} &= a_2^{\alpha a_1^+ a_1} a_2 a_2^{-\alpha a_1^+ a_1} = a_2 \end{aligned} \tag{21}$$

These two transformation leads to the two different $su(2)$ and two different $su(1,1)$ realization in one dimensions and the corresponding realizations are useful to study QES systems.

DIFFERENTIAL REALIZATIONS OF THE SU(2) AND SU(1,1) ALGEBRAS

The realizations (2) and (8) can be transformed in the form of the one dimensional differential equations in the Bargmann-Fock space when the boson operators realized as

$$a_1 = \frac{d}{dx}, \quad a_1^+ = x. \tag{22}$$

We can obtain two different differential realizations of the $su(2)$ and $su(1,1)$ algebras, depending on the choice of the α, in the equations (14) and (17).

$su(2)$ Realization

When the generators (2) of $su(2)$ algebra is transformed by using the similarity transformation operator S in the case of $\alpha = 1$, takes the form:

$$\begin{aligned} J'_+ &= SJ_+S^{-1} = -(a_1^+)^2 a_1 + N' a_1^+ \\ J'_- &= SJ_-S^{-1} = a_1 \\ J'_0 &= SJ_0S^{-1} = a_1^+ a_1 - \frac{N'}{2} \\ N' &= SNS^{-1} = a_2^+ a_2. \end{aligned} \tag{23}$$

These representations are called Gelfand-Dyson representation of the $su(2)$ algebra. Similarly we can easily obtain a second realization by using the transformation operator T and $\alpha = -1$. In this case the realization of $su(2)$ is given by

$$\begin{aligned} J'_+ &= TJ_+T^{-1} = a_1^+ \\ J'_- &= TJ_-T^{-1} = -a_1^+(a_1)^2 + a_1(N'-1) \\ J'_0 &= TJ_0T^{-1} = a_1^+a_1 - \frac{N'}{2} \\ N' &= TNT^{-1} = a_2^+a_2. \end{aligned} \tag{24}$$

The difference between the Schwinger and Gelfand-Dyson representation is that while in the first the total number of a_1 and a_2 bosons characterize the the system, in the later it is only the number of a_2 bosons that characterize the system. According to (8) the representation is characterized by a fixed number $2j$. Therefore in the Gelfand-Dyson representation, the primed generators can be expressed in terms of one boson operator a_1. According to the (7) takes the values $N' = 2j$. The realizations (23) and (24)are the well known generators of the $sl(2,R)$ algebra, in the Bargmann-Fock space, which play an important role in the quasi-exact solution of the Schrödinger equation. The linear and bilinear combinations of the generators form a second order differential equation and according to the Turbiner theorem the linear and bilinear combinations of the generators J'_i $(i = +, -, 0)$ are QES. The basis function of the primed generators of the $su(2)$ algebra is the degree of polynomial of order $2j$,

$$P_n(x) = (x^0, x^1, \cdots, x^{2j}). \tag{25}$$

su(1,1) realizations

By using the similar arguments given in previous subsection we can obtain two different one dimensional differential realization for the $su(1,1)$ algebra in the Bargmann-Fock space. The generators of the $su(1,1)$ algebra under the transformation of the T when $\alpha = 1$ takes the form

$$\begin{aligned} K'_+ &= TK_+T^{-1} = (a_1^+)^2a_1 + (L'+1)a_1^+ \\ K'_- &= TK_-T^{-1} = a_1 \\ K'_0 &= TK_0T^{-1} = a_1^+a_1 + \frac{L'+1}{2} \\ L' &= TLT^{-1} = a_2^+a_2. \end{aligned} \tag{26}$$

The other realization can be obtained by transforming the generators with the transformation operator S and choosing $\alpha = -1$:

$$\begin{aligned} K'_+ &= TK_+T^{-1} = a_1^+ \\ K'_- &= TK_-T^{-1} = a_1^+(a_1)^2 + (L'+1)a_1 \\ K'_0 &= TK_0T^{-1} = a_1^+a_1 + \frac{L'+1}{2} \\ L' &= TLT^{-1} = a_2^+a_2. \end{aligned} \tag{27}$$

The basis function of these generators are polynomials in x, in the Bargmann-Fock space. In the representations (26) and (27) the operator L' characterize the system, $L' = -2\ell - 1$.

Consequently the (quasi)exact solution of the two-mode bosonic Hamiltonians which include linear and bilinear combinations of the $su(2)$ or $su(1,1)$ algebra can be obtained by a suitable transformation.

APPLICATIONS

In this section we discuss the applicability of the method to solve the Hamiltonians of the various physical systems.

Karassiov-Klimov Hamiltonian

The method developed in this article can be used to obtain the solution of the various two-boson Hamiltonians. Consider the following family of Karassiov-Klimov[3] Hamiltonians

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \kappa a_1^{+s} a_2^r + \bar{\kappa} a_1^s a_2^{+r} \tag{28}$$

where $0 < r < s$, ω_1 and ω_2 are frequencies of two independent harmonic oscillator and κ and $\bar{\kappa}$ are coupling constants. The Hamiltonian was studied in the context of nonlinear Lie algebras[20], for $r = s = 2$ and $r = 1, \quad s = 2$. Let us consider the transformation of the Hamiltonian H by the transformation operator S, choosing $\alpha = r/s$,

$$H' = SHS^{-1} = \omega_1 a_1^+ a_1 + \omega_2 (a_2^+ a_2 - \frac{r}{s} a_1^+ a_1) + \kappa a_1^{+s} (a_2^+ a_2 - \frac{r}{s} a_1^+ a_1)^r + \bar{\kappa} a_1^s \tag{29}$$

Using the relation (7) and Bargmann-Fock space realizations (22)of the a_1 and a_1^+ the Hamiltonian takes the form

$$H' = (\omega_1 - \frac{r}{s}) x \frac{d}{dx} + 2j\omega_2 + \kappa x^s (2j - \frac{r}{s} x \frac{d}{dx})^r + \bar{\kappa} \frac{d^s}{dx^s} \tag{30}$$

The eigenvalue problem can be written as

$$H' P_n(x) = E P_n(x). \tag{31}$$

With the basis function (25) the eigenvalue equation (31) leads to the following recurrence relation

$$(\omega_1 - \frac{r}{s} - E + 2j\omega_2) P_n(E) + \kappa (2j - \frac{r}{s} n)^r P_{n+s}(E) + \bar{\kappa} \frac{n!}{(n-s)!} P_{n-s}(E) = 0 \tag{32}$$

and the wave function of the Hamiltonian H can be written as

$$\psi(x) = S^{-1} \sum_{k=0}^{2j} P_k(E) x^k \tag{33}$$

The wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then produced by the roots of such polynomials. If the E_k is a root of the polynomial $P_{k+1}(E)$, the series (22) terminates at $k > 2j\frac{s}{r}$ and E_k belongs to the spectrum of the corresponding Hamiltonian. The eigenvalues are then obtained by finding the roots of such polynomials.

The Hamiltonian (28) have been considered in[20], in the context of the nonlinear algebras, for the specific values of $s = r = 2$ and $s = 2, r = 1$. The Hamiltonian (23) can be expressed in terms of the generators of $su(2)$ algebra when $s = r = 2$

$$H = (\omega_1 - \omega_2) J_0 + \kappa J_+^2 + \bar{\kappa} J_-^2 + \frac{1}{2}(\omega_1 + \omega_2) N \tag{34}$$

and the transformed Hamiltonian can be cast into a differential operator

$$H = (\bar{\kappa} + \kappa x^4) \frac{d^2}{dx^2} + x(\omega_1 - \omega_2 + 2\kappa(3 - 2j) x^2) \frac{d}{dx} - 2j(\omega_1 + \kappa(1 - 2j) x^2), \tag{35}$$

by using the realization given in (23). Our result is coincides with the result given in[20]. The recurrence relations, with the basis function (25) is given by

$$\begin{aligned} \kappa(2j - n)(2j - n - 1) P_{n+2}(\lambda) + \bar{\kappa} n(n-1) P_{n-2}(\lambda) & \\ + (\omega_1 (n - 2j) - \omega_2 n - E) P_n(\lambda) &= 0 \end{aligned} \tag{36}$$

The Hamiltonians of the second and the third harmonic generation are the special case of the Karassiov-Klimov Hamiltonian and they are given by

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \kappa (a_1^{+2} a_2 + a_1^2 a_2^+) \tag{37a}$$
$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \kappa (a_1^{+3} a_2 + a_1^3 a_2^+) \tag{37b}$$

respectively. It is obvious that (28) can be put in the form of the (37a-b) and they can be expressed as one variable differential equations as in(30). Their eigenvalues and eigenfunctions can be obtained by using the recurrence relation (32) and (32), respectively.

Quantal system under thermal effect

The next example is the Hamiltonian of the quantal system under thermal effect[21]. In order to investigate thermal effects in quantum many-particle systems the Hamiltonian can be formulated by using two bosons creation and annihilation operators. The Hamiltonians of the symmetric and asymmetric rigid rotators have almost the same structure and their algebraic structure is $su(1,1)$ algebra. The Hamiltonians of such systems can be expressed as[22]:

$$H = \hbar\omega(a_1^+ a_1 - a_2^+ a_2) - i\gamma\hbar(a_1^+ a_2^+ - a_1 a_2) \tag{38}$$

Using the realizations (8) of $su(1,1)$ algebra we can rewrite the Hamiltonian:

$$H = \hbar\omega L - i\gamma\hbar(K_+ - K_-) \tag{39}$$

The Hamiltonian can be expressed as one dimensional differential equation by using the transformation operators S and T:

$$H' = SHS^{-1} = \hbar\omega(-2j-1) - i\gamma\hbar((x^2-1)\frac{d}{dx} - 2jx) \tag{40a}$$

$$H' = THT^{-1} = \hbar\omega(-2j-1) - i\gamma\hbar(x - x\frac{d^2}{dx^2} + 2j\frac{d}{dx}) \tag{40b}$$

respectively. According to the Turbiner theorem[7] this equation is exactly solvable.

Two variable differential equations of the special functions

Let us consider the following bosonic operator

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \alpha_1 a_1^+ a_2^+ + \alpha_2 a_1 a_2 - \frac{1}{2}a_1^2 a_2^2. \tag{41}$$

In terms of the generators of $su(1,1)$ the Hamiltonian can be expressed as

$$H = (\omega_1 + \omega_2)(K_0 - \frac{1}{2}) + \frac{1}{2}(\omega_1 - \omega_2)L + \alpha_1 J_+ + \alpha_2 J_- - \frac{1}{2}J_-^2 \tag{42}$$

The transformed Hamiltonian can be expressed as a differential operator

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + (\alpha_2 + x(\omega_1 + \omega_2 + \alpha_1 x))\frac{d}{dx} - (2k\alpha_1 x + \omega_1(2k+1)), \tag{43}$$

which can be written in the form of an eigenvalue equation

$$HR(x) = ER(x). \tag{44}$$

The solution of (43) can be obtained by introducing the wave function

$$R(x) = e^{-\int W(x)dx}\psi(x) \tag{45}$$

where the weight function $W(x)$ is given by

$$W(x) = -\alpha_2 - (\omega_1 + \omega_2)x - \alpha_1 x^2) \tag{46}$$

With the wave function given in (45) the Hamiltonian (43) can be transformed in the form of Schrödinger equation with the potential

$$V(x) = \frac{1}{2}(\alpha_2^2 - \omega_1(4k+3) - \omega_2) + (\alpha_2(\omega_1+\omega_2) - \alpha_1(2k+1))x + \frac{1}{2}(\alpha_1\alpha_2 + (\omega_1+\omega_2)^2)x^2 + \alpha_1(\omega_1+\omega_2)x^3 + \frac{\alpha_1^2}{2}x^4. \quad (47)$$

Therefore the bosonic operator given by (41) is related to the anharmonic oscillator potential. It is known that the Schrödinger equation with the potential given in(47) is QES. Let us turn our attention to the Hamiltonian (43). When we set

$$\alpha_1 = \alpha_2 = 0, \omega_1 = \omega_2 = 1/2, E = \frac{1}{2}(n - 2k - 1) \quad (48)$$

then the eigenvalue problem can be written as

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + x\frac{d}{dx} - \frac{n}{2}\right)R(x) = 0 \quad (49)$$

and the bosonic Hamiltonian takes the form

$$H = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 - a_1^2 a_2^2). \quad (50)$$

It is obvious that differential equation of the two dimensional Hermite polynomials in the Bargmann-Fock space can be expressed as:

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial x_1 \partial x_2} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right)R(x_1, x_2) = 0. \quad (51)$$

Our last example is the two-boson realization of the Schrödinger equation with the sextic harmonic oscillator potential. Let us consider the following bosonic operator:

$$L = (\omega_1 + \alpha_1)a_1^+ a_1 + (\omega_1 - \alpha_1)a_2^+ a_2 + \alpha_+ a_1^+ a_2^+ + (\alpha_- - \frac{1}{2})a_1 a_2 - \frac{1}{2}(a_2^+ a_1 a_2^2 + a_1^+ a_2 a_1^2) + \omega_1. \quad (52)$$

In terms of the generators of the $su(1,1)$ algebra it can be written as

$$H = 2\omega_1 J_0 + \alpha_+ J_+ + \alpha_- J_- + \alpha_1 N - J_0 J_-, \quad (53)$$

which can be converted to the differential operator

$$H = -x\frac{d^2}{dx^2} + (\alpha_- + k + x(2\omega_1 + \alpha_+ x))\frac{d}{dx} - 2k(\alpha_+ + (\omega_1 + \alpha_+ x)). \quad (54)$$

Let us change the variable and redefine the wave function for the eigenvalue problem

$$H\psi(x) = E\psi(x), x = \left(\frac{z}{2}\right)^2, \psi(z) = e^{\int W(z)dz}R(z) \quad (55)$$

where the weight function $W(z)$ is given by

$$W(z) = \frac{\alpha_+}{16}z^3 + \frac{\omega_1}{2}z + \frac{1 + 2\alpha_- + 2k}{2z}. \quad (56)$$

Upon this substitution the potential term in the Hamiltonian can be written as:

$$V(x) = \alpha_-\omega_1 - k(2\alpha_1+\omega_1) + \frac{(1+2\alpha_- + 2k)(3+2\alpha_- + 2k)}{4z^2} + \frac{\alpha_+(\alpha_- - 3k - 1) + 2\omega_1^2}{8} z^2 + \frac{\alpha_+\omega_1}{16} z^4 + \left(\frac{\alpha_+}{16}\right)^2 z^6 \quad (57)$$

This is the radial sextic oscillator potential studied in the literature.

CONCLUSION

In this paper we have discussed solution of the two-boson Hamiltonians and applications in physics. It was shown that the solution of the two-boson Hamiltonians can be transformed in the form of one-dimensional QES differential equation by applying a suitable similarity transformation. This transformation also leads to the connection between one and two-dimensional special functions of the physics.

The method given here is useful to study nonlinear quantum optical systems. The range of the Hamiltonian can be extended by the Bugoliov transformation of the boson operators. It is expected that this work will leads to the construction of the multi-boson QES Hamiltonians and their extensions to the matrix Hamiltonians. We have been presented a first step towards an extension of the quasi-exact solution of the multi-bosonic Hamiltonians.

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